The Exact Dimensions of a Family of Rectangular Coaxial Lines with Given Impedance

HENRY J. RIBLET, FELLOW, IEEE

Abstract—**The potential problem associated with a capacitor bounded by concentric rectangles may be solved by conformally mapping a certain L-shaped region onto the upper half plane. In certain cases, the integral, by which the Schwarz–Christoffel transformation may be expressed, can be evaluated in terms of two elliptic integrals of the first kind. Then the odd- and even-mode impedance of the related transmission line is readily found.**

Introduction

RECENTLY, Conning [1] has drawn attention to a solution to this problem for the special case of square concentric conductors given by Bowman [2], [3]. Now, Bowman used the fact that for concentric square conductors, the integral associated with the Schwarz–Christoffel transformation is a degenerate hyperelliptic integral that can always be expressed as the sum of two elliptic integrals of the first kind. There is, however, for a given characteristic impedance, a one parameter family of concentric rectangular conductors, containing the square conductors as a special case, whose dimensions may also be found in this way.

Transformation of the Integral

We are concerned with mapping the upper half of the u plane onto the region of the z plane bounded by $OABCDE$ as shown in Fig. 1 provided by the transformation

$$Z = \int_0^u \frac{M\sqrt{u}\,du}{\sqrt{(1-u)(1+\alpha u)(1+\beta u)(1-\alpha\beta u)}}. \quad (1)$$

Following Cayley [4], we introduce two new parameters:

$$k = (\sqrt{\beta} - \sqrt{\alpha})/\Delta \qquad \lambda = (\sqrt{\beta} + \sqrt{\alpha})/\Delta$$
$$k' = (1 + \sqrt{\alpha\beta})/\Delta \qquad \lambda' = (1 - \sqrt{\alpha\beta})/\Delta \quad (2)$$

with $\Delta = \sqrt{(1+\alpha)(1+\beta)}$. Of course, $k^2 + k'^2 = \lambda^2 + \lambda'^2 = 1$, and it is not difficult to invert (2) and find

$$\alpha = \left(\frac{\lambda - k}{\lambda' + k'}\right)^2 \qquad \beta = \left(\frac{\lambda + k}{\lambda' + k'}\right)^2. \quad (3)$$

If we now make the substitution

$$\sqrt{u} = \frac{(\lambda' + k')t}{\sqrt{1-\lambda^2 t^2} + \sqrt{1 - k^2 t^2}} \quad (4)$$

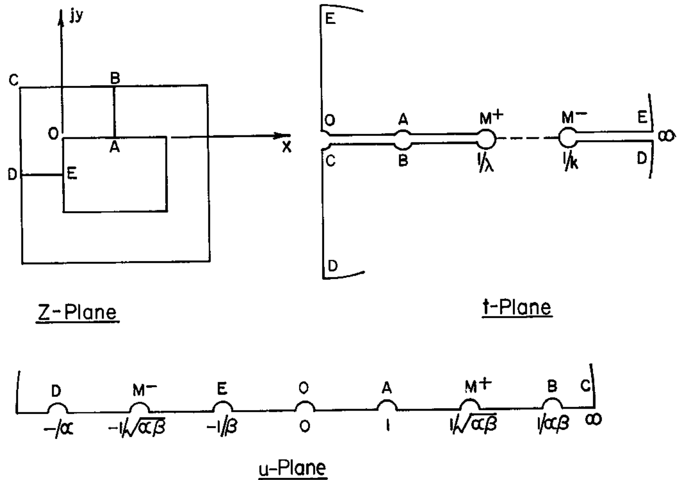


Fig. 1. Z, t, and u planes.

and write $\bar{L} = \sqrt{1-\lambda^2 t^2}$ and $\bar{K} = \sqrt{1 - k^2 t^2}$, one may readily follow Cayley [4] and find that

$$(1 + \alpha u)(1 + \beta u) = 4/(\bar{L} + \bar{K})^2 \quad (5)$$

and

$$(1 - u)(1 - \alpha\beta u) = 4(1 - t^2)/(\bar{L} + \bar{K})^2. \quad (6)$$

Moreover,

$$\sqrt{u}du = 2(\lambda' + k')^3 t^2\, dt/\bar{L}\cdot\bar{K}\cdot(\bar{L} + \bar{K})^3 \quad (7)$$

so that the integrand in (1) may be written:

$$\frac{(\lambda' + k')^2(\bar{K} - \bar{L})}{2(k' - \lambda')\sqrt{1-t^2}\cdot\bar{K}\cdot\bar{L}}$$

since $\bar{L}^2 - \bar{K}^2 = (\lambda'^2 - k'^2)t^2$. Thus

$$Z = M' \int_0^t \left\{ \frac{dt}{\sqrt{(1-t^2)(1-\lambda^2 t^2)}} - \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \right\}. \quad (8)$$

Transformation of the Path of Integration

It will be found that by squaring twice, (4) can be inverted and t can be expressed as

$$t^2 = \frac{(1+\alpha)(1+\beta)u}{(1+\alpha u)(1+\beta u)}. \quad (9)$$

Manuscript received October 25, 1971; revised December 29, 1971.

The author is with the Microwave Development Laboratories, Inc., Needham Heights, Mass. 02194.

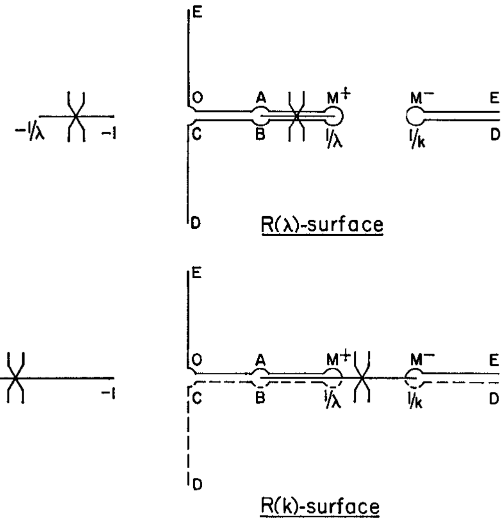


Fig. 2. Paths of integration.

Equations (8) and (9) then permit one to determine the relationship between points in the u plane and z plane in terms of well-known functions.

Equation (9) maps the upper half of the u plane into the right half of the t plane where the points corresponding to $OABCDE$ are as noted in Fig. 1. In addition, the critical points of (9) play an important part in the transformation. They occur for $u^2=1/\alpha\beta$ where $dt/du=0$. We notice that they correspond to values of $t=1/\lambda$ and $t=1/k$, while A and B both correspond to $t=1$. The problem of evaluating (1) as u traces out the real axis in the u plane while avoiding the points OAM^+BCDM^-E by means of small semicircles then becomes the problem of evaluating (8), while t traces out the boundary of the left half t plane avoiding the points O–E as shown in Fig. 1.

Now the integrands of the elliptic integrals in (8) are not single-valued functions of t in the t plane. In order to obtain unambiguous values for the integrals, it is convenient to plot the path of integration on the two Riemann surfaces where the integrands are single valued. If we denote $R(\lambda)=\sqrt{(1-t^2)(1-\lambda^2t^2)}$, then on the $R(\lambda)$ surface of Fig. 2, $R(\lambda)$ is a single-valued function of t because of the branch cuts between $-1/\lambda$ and -1 and 1 and $1/\lambda$. The same is true, of course, for $R(k)$ as a function of t on the $R(k)$ surface of Fig. 2. Moreover, the path of integration to be followed on both Riemann surfaces is shown in Fig. 2. For the left-hand integral of (8), the path of integration is entirely in the upper sheet since no branch cuts are crossed and we agree to start in the upper sheet; however, for the right-hand integral, half of the path is in the lower sheet, denoted by a dashed line.

The Integration

Adopting the convention that in the upper half plane of the upper sheet

$$\int_0^1 dt/R(\lambda) = K(\lambda)$$

we also have

$$\int_0^{1/\lambda} dt/R(\lambda) = jK'(\lambda) \qquad \int_\infty^{1/\lambda} dt/R(\lambda) = +K(\lambda)$$

$$\int_0^{j\infty} dt/R(\lambda) = jK'(\lambda).$$

Then

$$OA = Z(A) = \int_0^1 dt/R(\lambda) - \int_0^1 dt/R(k)$$
$$= K(\lambda) - K(k)$$

$$AB = \int_1^{1/\lambda} dt/R(\lambda) + \int_{1/\lambda}^1 dt/R(\lambda) - \int_1^{1/\lambda} dt/R(k)$$
$$- \int_{1/\lambda}^1 dt/R(k) = 2jK'(\lambda)$$

$$BC = \int_1^0 dt/R(\lambda) - \int_1^0 dt/R(k) = -K(\lambda) - K(k)$$

$$CD = \int_0^{-j\infty} dt/R(\lambda) - \int_0^{-j\infty} dt/R(k)$$
$$= -jK'(\lambda) - jK'(k)$$

$$DE = \int_\infty^{1/k} dt/R(\lambda) + \int_{1/k}^\infty dt/R(\lambda) - \int_\infty^{1/k} dt/R(k)$$
$$- \int_{1/k}^\infty dt/R(k) = 2K(k)$$

$$EO = \int_{j\infty}^0 dt/R(\lambda) - \int_{+j\infty}^0 dt/R(k)$$
$$= -jK'(\lambda) + jK'(k). \tag{10}$$

In making these determinations, the integrals are line integrals and the precise paths of integration, as shown in Fig. 2, are taken into account. Moreover, $R(\lambda)$ changes sign in going from a point in the upper sheet to the point just below it in the lower sheet or in crossing a branch cut in the same sheet.

The Geometry

Now in terms of real positive quantities when directed upward or to the right,

$$\overline{OA} = K(\lambda) - K(k) \qquad \overline{DC} = K'(\lambda) + K'(k)$$
$$\overline{AB} = 2K'(\lambda) \qquad \overline{DE} = 2K(k)$$
$$\overline{CB} = K(\lambda) + K(k) \qquad \overline{EO} = K'(k) - K'(\lambda). \tag{11}$$

Thus given any two numbers $0<\alpha<\beta$ whose product $\alpha\beta<1$, we may in turn determine two numbers $k<\lambda<1$ that are the moduli of the complete elliptic integrals of the first kind determining the dimensions of concentric rectangles according to (11). Moreover these rectangles are always realizable since $K(\lambda)$ is monotonically increasing and $K'(\lambda)$ is monotonically decreasing with λ.

The Characteristic Impedance

It now remains to determine the capacity per unit length of the structure. It is well known that total capacity is invariant under conformal transformation so we have to determine the upper half-plane capacity of section EA with respect to section BD (in the u plane of Fig. 1). This may be accomplished by transforming the upper half u plane into a rectangle by the transformation $t = sn^2 w$ [3, pp. 58 and 59]. Then the upper half-plane capacity C is

$$C = \frac{K'(k_o)}{K(k_o)}$$

where

$$k_o^2 = \frac{(b-a)(d-c)}{(d-b)(c-a)}$$

with $a = -1/\alpha$, $b = -1/\beta$, $c = 1$, and $d = 1/\alpha\beta$. So,

$$k_o^2 = \frac{(\beta - \alpha)(1 - \alpha\beta)}{\beta(1+\alpha)^2}. \tag{12}$$

The characteristic impedance Z_0 of the transmission line is then given by

$$Z_0 = \frac{376.7\, K(k_o)}{4\, K'(k_o)} = 94.18\, K(k_o)/K'(k_o). \tag{13}$$

Now the two parameters α and β will not determine all concentric rectangular coaxial sections, but if we specify the characteristic impedance, (13) determines k_o and (12) then defines a one parameter family of rectangular coaxial lines with the given characteristic impedance. If $\beta = 1$, the rectangles become concentric squares and the solution of Bowman results. Since $k_o'^2 = 1 - k_o^2$, (12) becomes

$$k_o'^2 = \frac{\alpha(1+\beta)^2}{\beta(1+\alpha)^2}. \tag{14}$$

Since $k_o' < 1$, it follows readily from (14) that for $\beta > 0$, there is always an α such that $0 < \alpha < \beta$ and, from (12), that $\alpha\beta < 1$. Thus in (12) or (14) β may be any positive number. However, it is no restriction to put $\beta \leq 1$ since β may be replaced by $1/\beta$ in (14) without changing α or $k_o'^2$. When this substitution is made in (2), k and λ' and k' and λ are interchanged with the effect that the rectangles in the z plane of Fig. 1 are rotated by 90°.

The capacitance of the structure when two of the opposing walls are magnetic will be useful later in determining its even-mode fringing capacitance. If the wall CD is to be magnetic, then in Fig. 1 we require the capacity between EA and BC. Now $a = \infty$, $b = -1/\beta$, $c = 1$, and $d = 1/\alpha\beta$. Then

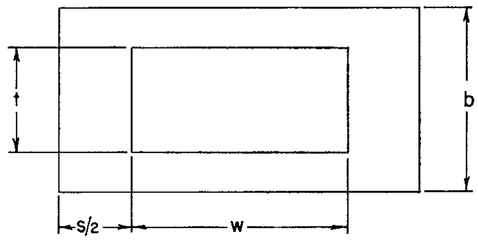
$$k_e^2 = \frac{1 - \alpha\beta}{1 + \alpha} \tag{15}$$

and the capacitance of one quadrant of the rectangle is $K'(k_e)/K(k_e)$.

One may specify, as above, the even-mode impedance of the structure. Then k_e is fixed and a one parameter family of concentric rectangular conductors can be found by selecting α and β to satisfy (15).

TABLE I

Dimensions and Fringing Capacities



Z_o	t/b	s/b	$w/(b-t)$	C_{fo}	C'_{fo}	C_{fe}	C'_{fe}
50	.40000	.59993	.65990	1.2237	1.2246	.68770	.68796
60	.40000	.58844	.33884	1.2309	1.2378	.67854	.68057
68	.40000	.53367	.11103	1.2740	1.3103	.63361	.64245

Applications

With the aid of a digital computer, it is easy to obtain the dimensions of the one parameter family of concentric rectangles with a given characteristic impedance. Then one of them can be selected for some desirable trait. For example, Table I gives s/b, t/b, and $w/(b-t)$ for three rectangular coaxial configurations with different values of characteristic impedance, but with the same value of t/b. It should be pointed out also that for these families, the inner conductor is nearly equispaced from the outer conductor.

This theory provides one with some additional information regarding the amount of interaction between the even- and odd-mode fringing capacitances calculated by Getsinger [5], for the case of an inner conductor of finite thickness. Table I gives values of the odd- and even-mode fringing capacities C_{fo} and C_{fe} using the exact theory beside approximate values C_{fo}' and C_{fe}' for the same s/b and t/b obtained from formulas similar to those used by Getsinger.

For $w/(b-t) = 0.11$, the error in C_{fo} is 2.85 percent, while the error in C_{fe} is 1.4 percent. This shows that Cohn's [6] criterion of $w/(b-t) \geq 0.35$ for C_f to be unaffected by proximity, is too cautious for equispaced conductors when $t > 0$. This is not surprising since

Cohn's criterion is based on the behavior of the fringing capacities for $t=0$ and one might expect intuitively that the interaction of fringing capacities would decrease markedly as t increases.

References

[1] S. W. Conning, "The characteristic impedance of square coaxial line," *IEEE Trans. Microwave Theory Tech.* (Corresp.), vol. MTT-12, p. 468, July 1964.

[2] F. Bowman, "Notes on two-dimensional electric field problems," *Proc. London Math. Soc.*, vol. 39, no. 2, pp. 211–214, 1935.

[3] ——, *Introduction to Elliptic Functions with Applications.* New York: Dover, 1961, pp. 39–104.

[4] A. Cayley, *Elliptic Functions.* New York: Dover, p. 360.

[5] W. J. Getsinger, "Coupled rectangular bars between parallel plates," *IRE Trans. Microwave Theory Tech.*, vol. MTT-10, pp. 65–72, Jan. 1962.

[6] S. B. Cohn, "Problems in strip transmission lines," *IRE Trans. Microwave Theory Tech.* (*Special Issue: Symposium on Microwave Strip Circuits*), vol. MTT-3, pp. 119–126, Mar. 1955.

Coupled Power Equations for Backward Waves

D. MARCUSE, MEMBER, IEEE

Abstract—**Two waves traveling in opposite directions that are coupled by a random coupling function are considered. These two waves can be described in a standard way by coupled wave equations. It is possible to derive coupled equations for the power carried by these two waves. The form of the coupled power equations differs depending on the assumptions that are made for the initial conditions. The validity of the coupled power equations has been confirmed by a computer-simulated experiment.**

Introduction

COUPLED POWER equations for waves traveling in opposite directions have been derived by Rowe [1] under the assumption that the coupling function has a white-noise spectrum and that the initial conditions for both waves have been specified at the far end of the transmission lines. He thus assumes that the output of mode (or line) 1 is specified at the end of the guide and that no power is incident at the far end in the reflected wave. His theory predicts the expected value of the reflected wave at the input of the line, as well as the expected values of the input waves that are required to obtain the fixed output value of the incident mode.

If one considered it as an established fact that the power exchange between the two waves can be treated by adding power instead of amplitude, one would write down intuitive coupled power equations that differ in form from the coupled power equations that Rowe derived. The question arises whether those intuitive equations are meaningless or how they are related to Rowe's equations. In order to gain insight into that problem, we conducted a computer-simulated experiment tracing waves through ten simulated waveguides with random coupling and compared the average output power obtained from the experiment with the prediction of the theories. The experiment can be done in several ways. It is possible to launch a constant amplitude into each of the ten random waveguides and to compute the average values of the power output of the incident wave at the far end of the guide, as well as the average power of the reflected wave at the near end of the guide. The result of this experiment agreed strikingly with the intuitive coupled power equations, while it was definitely at odds with Rowe's equations. However, the experimental conditions did not conform to Rowe's assumptions. We then changed the conditions requiring that the output voltage of the incident wave have a fixed value at the far end while no power enters the reflected mode at the far end. The experimental values now showed far larger scatter than in the first case, but comparison indicated that they were in agreement with Rowe's equations while they definitely contradicted the predictions of the intuitive equations if they were applied to this case.

The result of this experiment points to the conclusion that different differential equations are required to describe the statistical outcome of coupled wave experiments in which the two waves travel in opposite directions. One set of equations describes the situation in which the input wave is known while no reflected wave is allowed to enter at the far end. Another set of coupled power equations describes the experimental situation in which we require that the output wave has a definite amplitude, while again no power is allowed to enter the reflected mode at the far end.

In this paper both types of coupled power equations are derived from the coupled wave equations using per-

Manuscript received November 8, 1971; revised February 15, 1972.
The author is with Bell Telephone Laboratories, Crawford Hill Laboratory, Holmdel, N. J. 07733.